

VISCOSITY SOLUTIONS, ALMOST EVERYWHERE SOLUTIONS AND EXPLICIT FORMULAS

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ABSTRACT. Consider the differential inclusion $Du \in E$ in $\mathbb{R}^n$. We exhibit an explicit solution that we call *fundamental*. It also turns out to be a *viscosity solution* when properly defining this notion. Finally, we consider a Dirichlet problem associated to the differential inclusion and we give an iterative procedure for finding a solution.

1. INTRODUCTION

Existence of *almost everywhere* solutions of the first order Dirichlet problem related to *implicit differential equations* of the type

$$(1) \quad \begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

has recently been extensively studied in the book [6] by the authors. Here $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function and we look for a Lipschitz-continuous solution $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$. A wide literature on this subject can be found in [6], not only for scalar problems such as this one, but also for *vector-valued solutions of first order systems* related to maps $u : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $F : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^N$, for some $m, N \geq 1$.

Existence of *viscosity solutions* of the Dirichlet problem (1) is now well established. It has been studied by many authors starting with Hopf, Lax, Kruzkov and Crandall-Lions; see for example [1] or [6] for more historical comments. One of the earliest and still one of the most complete monographs on the subject is [10] by P.L. Lions. The research in this field remains very active; in particular H. Ishii and P. Loreti [8], motivated by an optimization problem, recently gave an existence result of viscosity solutions of the Dirichlet problem (1). See also [2] and [9].

In this paper we give some existence results, either in the case of *almost everywhere* solutions, or, when possible, of *viscosity solutions*. One of our aims is to give some constructive explicit formulas (cf. Theorems 1 and 6). Moreover, if the geometry of the set Ω and the assumptions on the function F make it possible, following [3] we give (cf. Corollary 8) an explicit formula for a viscosity solution of the Dirichlet problem (1), simply in terms of sup and inf. Otherwise, with general F and Ω , we propose, in Section 4, an iteration scheme for characterizing a solution.

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In Section 2 we introduce the notion of viscosity solution of a *differential inclusion*; namely: given a closed set E , we say that a function u is a viscosity solution of the differential inclusion

$$(2) \quad Du(x) \in E, \quad x \in \Omega,$$

if u is a viscosity solution of the equation

$$(3) \quad F(Du(x)) = 0, \quad x \in \Omega,$$

where $F(\xi) = \text{dist}\{\xi, E\}$. We will prove in Theorem 6 that the function $L : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by

$$L(x) = \max\{\langle \xi, x \rangle : \xi \in E\},$$

is a viscosity solution of the differential inclusion (2), i.e. it is a *fundamental solution* of the equation (3).

2. FUNDAMENTAL SOLUTION AND VISCOSITY SOLUTIONS OF DIFFERENTIAL INCLUSIONS

We start by recalling some classical definitions and notations in convex analysis. We say that $\xi \in \mathbb{R}^n$ is an *extreme point* for a convex set $K \subset \mathbb{R}^n$ if the conditions

$$\begin{cases} \xi = t\xi_1 + (1-t)\xi_2, \\ \xi_1, \xi_2 \in K, \quad t \in (0, 1), \end{cases}$$

imply that $\xi = \xi_1 = \xi_2$.

If E is a set (not necessarily convex) of $\mathbb{R}^n$, we denote by E_{ext} the set of extreme points of the convex hull of E denoted by $\text{co } E$ (note that $E_{\text{ext}} \subset E$).

We also recall that the domain of a convex function $L : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined as

$$\text{dom } L = \{x \in \mathbb{R}^n : L(x) < +\infty\}.$$

Theorem 1 below generalizes an analogous result obtained by Ischii and Loreti (see the proof of Theorem 2.2 in [8]) in the case that E is the level set of a continuous, positively homogeneous function of degree one, equal to zero only at the origin of $\mathbb{R}^n$.

Theorem 1. *Let E be a compact set of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ let*

$$L(x) = \max\{\langle \xi, x \rangle : \xi \in E\}.$$

Then

$$DL(x) \in E \quad \text{a.e. } x \in \mathbb{R}^n.$$

Remark 2. (i) It should be noted that in fact the theorem is more precise, namely

$$DL(x) \in \overline{E_{\text{ext}}} \subset E \cap \partial \text{co } E \quad \text{a.e. } x \in \mathbb{R}^n.$$

(ii) If E is any set, not necessarily closed or bounded, then the proof gives (replacing max by sup) that

$$DL(x) \in \overline{E} \quad \text{a.e. } x \in \text{dom } L.$$

(iii) In terms of convex analysis and anticipating on (4) we can say that L is the support function of $\text{co } E$.

Before proceeding with the proof it might be interesting to rewrite the theorem in terms of equations.

Corollary 3. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function such that*

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}$$

is a bounded set. Let $L(x) = \max\{\langle \xi, x \rangle : F(\xi) = 0\}$; then

$$F(DL(x)) = 0 \text{ a.e. } x \in \mathbb{R}^n.$$

Proof. The following representation formula for L holds (see Rockafellar [12], Theorem 32.2):

$$(4) \quad L(x) = \max\{\langle \xi, x \rangle : \xi \in \text{co } E\} = \max\{\langle \xi, x \rangle : \xi \in E\}, \quad \forall x \in \mathbb{R}^n.$$

In fact one has the more precise result (see Rockafellar [12], Corollary 32.3.2)

$$(5) \quad L(x) = \max\{\langle \xi, x \rangle : \xi \in \text{co } E\} = \max\{\langle \xi, x \rangle : \xi \in E_{\text{ext}}\}, \quad \forall x \in \mathbb{R}^n.$$

Let $\{\xi_h\}_{h \in \mathbb{N}}$ be a (finite or) countable dense subset of $E_{\text{ext}} \subset E$ and, analogously to (4), for every $h \in \mathbb{N}$ and for every $x \in \mathbb{R}^n$ let us define

$$L_h(x) = \max\{\langle \xi_1, x \rangle, \langle \xi_2, x \rangle, \dots, \langle \xi_h, x \rangle\}.$$

Clearly the gradient DL_h exists almost everywhere in $\mathbb{R}^n$ and

$$(6) \quad DL_h(x) \in \{\xi_1, \xi_2, \dots, \xi_h\} \subset E_{\text{ext}}, \quad \text{a.e. } x \in \mathbb{R}^n.$$

For every $x \in \mathbb{R}^n$ the sequence $L_h(x)$ is increasing with respect to $h \in \mathbb{N}$ and we have

$$L(x) = \sup\{L_h(x) : h \in \mathbb{N}\} = \lim_{h \rightarrow +\infty} L_h(x).$$

For every $h \in \mathbb{N}$ the sequence $L_h(x)$ is convex with respect to $x \in \mathbb{R}^n$ and

$$\text{dom } L_h = \text{dom } L = \mathbb{R}^n.$$

Thus we can apply Lemma 4 and we obtain that, at every point where L_h and L are differentiable (i.e., almost everywhere in $\mathbb{R}^n$),

$$DL_h(x) \rightarrow DL(x).$$

Therefore, by (6), we get the conclusion

$$DL(x) \in \overline{E_{\text{ext}}} \subset E, \quad \text{a.e. } x \in \mathbb{R}^n.$$

□

In the proof of Theorem 1 we used a result given in [11] (Lemma 5.9), that we recall here in a form more appropriate to the applications given in this paper.

Lemma 4. *Let $\{L_h\}_{h \in \mathbb{N}}$ be a sequence of convex functions, defined on $\mathbb{R}^n$ with values on $\mathbb{R} \cup \{+\infty\}$, with pointwise limit $L : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$. At every point $x \in \text{int}[(\bigcap_{h \in \mathbb{N}} \text{dom } L_h) \cap \text{dom } L]$, where L_h and L are differentiable, the gradient $DL_h(x)$ converges in $\mathbb{R}^n$ to the gradient $DL(x)$.*

Proof. For every $h \in \mathbb{N}$ let $\text{dom } L_h$ and $\text{dom } L$ be the domains of L_h and L . Then each L_h is locally Lipschitz-continuous in $\text{int } \text{dom } L_h$ and L is locally Lipschitz-continuous in $\text{int } \text{dom } L$. Therefore, for every $h \in \mathbb{N}$, there exists a set $N_h \subset \text{dom } L_h \subset \mathbb{R}^n$ of zero measure such that L_h is differentiable at every point of $\text{dom } L_h \setminus N_h$. Analogously, there exists a set $N \subset \text{dom } L$ of zero measure such that

L is differentiable at every point of $\text{dom } L \setminus N$. Then the set of points $x \in \mathbb{R}^n$ where L_h and L are differentiable is (possibly empty and) given by

$$\left(\bigcap_{h \in \mathbb{N}} (\text{dom } L_h \setminus N_h) \right) \cap (\text{dom } L \setminus N)$$

and differs from the intersection of their domains $(\bigcap_{h \in \mathbb{N}} \text{dom } L_h) \cap \text{dom } L$ by (at most) a set $(\bigcup_{h \in \mathbb{N}} N_h) \cup N$ of zero measure. Let $x \in \text{int}[(\bigcap_{h \in \mathbb{N}} \text{dom } L_h) \cap \text{dom } L]$ be a point of $\mathbb{R}^n$ where L_h and L are differentiable. Let $i \in \{1, 2, \dots, n\}$ and $h \in \mathbb{N}$ be fixed. Then at $x = (x_1, \dots, x_i, \dots, x_n)$ the partial derivatives $\partial L_h / \partial x_i$ and $\partial L / \partial x_i$ are well defined. An elementary application of the convex inequality for the function L_h gives the monotonicity of the difference quotient; precisely, if $t > 0$ is sufficiently small and if, as usual, we denote by $x \pm te_i$ the two points of $\mathbb{R}^n$ with coordinates respectively $(x_1, \dots, x_{i-1}, x_i \pm t, x_{i+1}, \dots, x_n)$, we have

$$\frac{L_h(x - te_i) - L_h(x)}{-t} \leq \frac{\partial L_h}{\partial x_i}(x) \leq \frac{L_h(x + te_i) - L_h(x)}{t}$$

and, in the limit as $h \rightarrow +\infty$,

$$\frac{L(x - te_i) - L(x)}{-t} \leq \liminf_{h \rightarrow +\infty} \frac{\partial L_h}{\partial x_i}(x) \leq \limsup_{h \rightarrow +\infty} \frac{\partial L_h}{\partial x_i}(x) \leq \frac{L(x + te_i) - L(x)}{t}.$$

Since L is differentiable at x , as $t \rightarrow 0^+$ we obtain that $\partial L_h / \partial x_i(x)$ converges to $\partial L / \partial x_i$. The property being such that for every $i \in \{1, 2, \dots, n\}$, we have the conclusion, i.e., that the gradient $DL_h(x)$ converges in $\mathbb{R}^n$ to the gradient $DL(x)$. $\square$

Remark 5. With a slightly different proof, as in Lemma 5.9 in [11], we can give a compactness result. Precisely, we can show that from every locally bounded sequence $\{L_h\}_{h \in \mathbb{N}}$ of convex functions ($\{L_h\}_{h \in \mathbb{N}}$ uniformly bounded in $L_{\text{loc}}^\infty(\Omega)$, with Ω open set in $\mathbb{R}^n$) it is possible to select a subsequence $\{L_{h_k}\}_{k \in \mathbb{N}}$ whose gradients $\{DL_{h_k}\}_{k \in \mathbb{N}}$ converge almost everywhere in Ω , and at the same time $\{L_{h_k}\}_{k \in \mathbb{N}}$ converges in the strong topology of $W_{\text{loc}}^{1,q}(\Omega)$, for every $q \in [1, +\infty)$.

With the help of the above construction we can give a definition of what we mean by viscosity solutions of differential inclusions. Given a closed set E , we say that a function u is a *viscosity solution of the differential inclusion*

$$Du(x) \in E, \quad x \in \mathbb{R}^n,$$

if u is a viscosity solution of the equation

$$F(Du(x)) = 0, \quad x \in \mathbb{R}^n,$$

where $F(\xi) = \text{dist}\{\xi, E\}$. We therefore have the following result.

Theorem 6. *Let E be a compact set of $\mathbb{R}^n$. For every $x \in \mathbb{R}^n$ let*

$$L(x) = \max\{\langle \xi, x \rangle : \xi \in E\}.$$

Then L is a viscosity solution of

$$DL(x) \in E, \quad x \in \mathbb{R}^n.$$

Proof. The function L being convex we have that $D^+L(x)$ (the superdifferential of L at x ; see [1] and [6] for the precise definition of this set) is either empty or reduced to $\{DL(x)\}$, i.e. x is a point of differentiability of L and we know by Theorem 1 that at such points $DL(x) \in E$. We therefore have that

$$F(p) = 0, \quad \forall p \in D^+L(x),$$

which means that L is a viscosity subsolution (see Proposition 4.7 of [6]) of $F(Du) = 0$.

Since $F \geq 0$ we deduce trivially that

$$F(p) \geq 0, \quad \forall p \in D^-L(x),$$

where $D^-L(x)$ is the subdifferential of L at x . This means that L is a viscosity supersolution of $F(Du) = 0$.

Combining these two results we have indeed that L is a viscosity solution of $F(Du) = 0$ and hence of $Du \in E$. $\square$

3. FUNDAMENTAL SOLUTION AND THE BOUNDARY CONDITION

We now want to discuss a Dirichlet problem in a bounded domain. We first fix the notations.

We let $\Omega \subset \mathbb{R}^n$ be a bounded open convex set and denote by $\nu(y)$ the outward unit normal at $y \in \partial\Omega$ (that exists at almost all points $y \in \partial\Omega$, since Ω is convex).

We next let $E \subset \mathbb{R}^n$ be a compact set with $0 \in \text{intco } E$. We then associate to $\text{co } E$ its *gauge* ρ , which is a convex and positively homogeneous of degree one function, such that

$$\text{co } E = \{\xi \in \mathbb{R}^n : \rho(\xi) \leq 1\}.$$

Recall also that

$$L(x) = \max \{\langle \xi, x \rangle : \xi \in E\}.$$

We should immediately note that, with our hypotheses on E (and invoking (4)), the function L is in fact the polar of ρ , denoted also sometimes by ρ^0 .

We finally consider the Dirichlet problem

$$\begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

We could also consider the case of a more general boundary datum of class C^1 but the analysis can then be carried in a straightforward manner.

We have the following theorem that is inspired by Cardaliaguet-Dacorogna-Gangbo-Georgy [3] (see also [6]).

Theorem 7. *Let Ω , ν , E , ρ and L be as above and satisfy in addition*

$$(7) \quad \frac{-\nu(y)}{\rho(-\nu(y))} \in E, \quad \text{a.e. } y \in \partial\Omega;$$

then the function $u : \mathbb{R}^n \rightarrow \mathbb{R}$, defined by

$$(8) \quad u(x) = \min \{L(x - y) : y \in \partial\Omega\},$$

solves the Dirichlet problem

$$(9) \quad \begin{cases} Du(x) \in E, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

As before we rewrite this theorem in terms of functions.

Corollary 8. *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous with $F(\xi) \rightarrow \infty$ as $|\xi| \rightarrow \infty$ and $F(0) < 0$. Set*

$$E = \{\xi \in \mathbb{R}^n : F(\xi) = 0\}.$$

Let Ω , ν , ρ and L be as above. If

$$(10) \quad F\left(\frac{-\nu(y)}{\rho(-\nu(y))}\right) = 0 \text{ a.e. } y \in \partial\Omega,$$

then

$$(11) \quad u(x) = \min\{L(x - y) : y \in \partial\Omega\}$$

solves

$$(12) \quad \begin{cases} F(Du(x)) = 0, & \text{a.e. } x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Furthermore if $E \subset \partial \operatorname{co} E$, then u is a viscosity solution.

Remark 9. (i) The first part of the corollary follows immediately from the theorem. The fact that u is a viscosity solution (when $E \subset \partial \operatorname{co} E$) was established in [3].

(ii) Note that if, in addition, $\partial \operatorname{co} E \subset E$ (which happens if, for instance, F is convex or more generally if the set $\{\xi : F(\xi) \leq 0\}$ is convex), then (10) is always satisfied. In fact, since ρ is positively homogeneous of degree one,

$$\rho\left(\frac{-\nu(y)}{\rho(-\nu(y))}\right) = 1 \Rightarrow \frac{-\nu(y)}{\rho(-\nu(y))} \in \partial \operatorname{co} E \subset E.$$

Moreover, if $E = \partial \operatorname{co} E$, u defined in (11) is the unique viscosity solution of (12).

(iii) According to Theorem 4.1 of Lions [10], the Dirichlet problem (12) always has a viscosity solution. However the solution given by (11) is not necessarily a viscosity solution; it is so when $E \subset \partial \operatorname{co} E$.

We can now proceed with the proof of the theorem.

Proof of Theorem 7. We recall the following two facts (the first one is just the Hopf-Lax formula and the second one is Lemma 2.9 in [3] or Lemma 4.17 in [6]). We also use the standard notation $D^+u(x)$, respectively $D^-u(x)$, for the superdifferential, respectively the subdifferential, of u at x (see [6] for more details).

Fact 1: The function u is the viscosity solution of

$$(13) \quad \begin{cases} \rho(Du(x)) = 1, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Fact 2: Let $y(x) \in \partial\Omega$ be such that

$$u(x) = L(x - y(x)).$$

Then, if $p \in D^-u(x)$ (i.e. $D^-u(x)$ is non empty), the outward unit normal $\nu(y(x))$ is well defined and there exists $\lambda(y(x)) > 0$ such that

$$(14) \quad p = -\lambda(y(x))\nu(y(x)).$$

Since we are interested in almost everywhere solutions we need only to consider points $x \in \Omega$ where $D^+u(x) = D^-u(x) = \{Du(x)\}$. Combining (13) and (14) with $p = Du(x)$ and the homogeneity of ρ , we get that $\lambda(y) = 1/\rho(-\nu(y))$ and hence

$$Du(x) = \frac{-\nu(y)}{\rho(-\nu(y))}.$$

The hypothesis (7) leads to the result $Du \in E$. □

4. THE ITERATION SCHEME

As above we let $\Omega \subset \mathbb{R}^n$ be a nonempty bounded open set. We want to find, with the help of the previous construction, a solution $u \in W_0^{1,\infty}(\Omega)$ of the differential inclusion

$$Du(x) \in E, \text{ a.e. } x \in \Omega,$$

where $E \subset \mathbb{R}^n$ is a compact set with $0 \in \text{intco } E$. We let ρ be the gauge associated to $\text{co } E$.

We will find a sequence of disjoint convex open sets $\Omega_i \subset \Omega$ so that

$$\text{meas} \left[\Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i \right] = 0$$

and the function u will be defined as

$$u(x) = \begin{cases} \inf \{L(x-y) : y \in \partial\Omega_i\}, & x \in \Omega_i, \\ 0, & x \in \Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i. \end{cases}$$

Observe that u is a viscosity solution of the Dirichlet problem $Du \in E$ in Ω_i , $u = 0$, on $\partial\Omega_i$ for every i (but not globally in Ω).

Any Vitali covering by level sets of the function L has all the above requirements. However we will choose, among them, one with some maximality properties. In particular we want that $\Omega_1 = \Omega$ if Ω is convex and $\frac{-\nu}{\rho(-\nu)} \in E$, a.e. on $\partial\Omega$, where ν is the outward unit normal to Ω (recall that this always happens if $E = \partial \text{co } E$ or if Ω is the level set of the function L).

Before describing this construction we need to introduce some notations.

Notation 10. Let $x_0 \in \mathbb{R}^n$. We let G_{x_0} be the set of all gauges centered at x_0 . In other words this is the set of all convex functions $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}$ satisfying

$$\begin{aligned} \gamma(x_0) = 0, \quad \gamma(x) > 0, \quad \forall x \in \mathbb{R}^n \setminus \{x_0\}, \\ \gamma(t(x-x_0) + x_0) = t\gamma(x), \quad \forall x \in \mathbb{R}^n, \quad \forall t > 0. \end{aligned}$$

Proposition 11. (i) If $\gamma \in G_{x_0}$ is differentiable at $x \in \mathbb{R}^n$ (this happens at almost all points), then it is differentiable at any $x_t \in \mathbb{R}^n$ of the form $x_t = t(x-x_0) + x_0$, $t > 0$ and

$$D\gamma(x_t) = D\gamma(x).$$

In particular γ is differentiable at almost all points of $\{x \in \mathbb{R}^n : \gamma(x) = 1\}$.

(ii) Let $C \subset \mathbb{R}^n$ be a nonempty bounded open convex set and $x_0 \in \text{int } C$. The gauge of C centered at x_0 is defined as

$$\gamma_{C,x_0}(x) = \inf \left\{ \lambda \geq 0 : x_0 + \frac{x-x_0}{\lambda} \in C \right\}.$$

Then $\gamma_{C,x_0} \in G_{x_0}$ and

$$\begin{aligned} C &= \{x \in \mathbb{R}^n : \gamma_{C,x_0}(x) < 1\}, \\ \partial C &= \{x \in \mathbb{R}^n : \gamma_{C,x_0}(x) = 1\}. \end{aligned}$$

Remark 12. At almost every point $x \in \partial C$, γ_{C,x_0} is differentiable and $D\gamma_{C,x_0}(x)$ is then an outward normal to C .

We will proceed inductively to define Ω_i . We start by choosing a sequence of points in Ω , $\{x^N\}_{N=1}^\infty$, dense in Ω . We set $\Omega_0 = \emptyset$ and assume that Ω_i has already been defined. If $\Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k = \emptyset$, then the procedure is already over. We then define, $N = N(i+1)$,

$$N(i+1) = \min \left\{ N : x^N \in \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right\}$$

and we label $x_{i+1} = x^{N(i+1)}$ (so that $x_1 = x^1$). We then choose $r_{i+1} > 0$ sufficiently small so that

$$\left\{ x \in \mathbb{R}^n : l_{r_{i+1}}(x) \equiv \frac{L(x_{i+1} - x)}{r_{i+1}} < 1 \right\} \subset \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k,$$

where

$$L(x) = \max \{ \langle \xi, x \rangle : \xi \in E \}.$$

This is always possible since $\Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k$ is an open set, $x_{i+1} \in \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k$, $L(0) = 0$ and L is locally Lipschitz.

We next define

$$\begin{aligned} & \Gamma \left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right) \\ &= \left\{ \begin{array}{l} \gamma \in G_{x_{i+1}} : \frac{-D\gamma(x)}{\rho(-D\gamma(x))} \in E, \text{ a.e. } x \in \mathbb{R}^n \\ \{x \in \mathbb{R}^n : l_{r_{i+1}}(x) < 1\} \subset \{x \in \mathbb{R}^n : \gamma(x) < 1\} \subset \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \end{array} \right\}. \end{aligned}$$

Note that $l_{r_{i+1}} \in \Gamma \left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right)$, since, by Theorem 1, $DL \in E$ and $\rho(DL) =$

1. Observe also that if $\gamma \in \Gamma \left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right)$, then

$$(15) \quad \gamma \leq l_{r_{i+1}}.$$

We now claim that there exists $\gamma_{i+1} \in \Gamma \left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right)$ such that if

$$\Omega_{i+1} = \{x \in \mathbb{R}^n : \gamma_{i+1}(x) < 1\},$$

then

$$\text{meas}(\Omega_{i+1}) = \sup_{\gamma \in \Gamma \left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right)} [\text{meas} \{x \in \mathbb{R}^n : \gamma(x) < 1\}].$$

Indeed let $\{\gamma^s\}$ be a maximizing sequence. From (15), we deduce that up to a subsequence, that we still label $\{\gamma^s\}$, the sequence converges to an element $\gamma_{i+1} \in \Gamma \left(x_{i+1}, \Omega \setminus \bigcup_{k=0}^i \overline{\Omega}_k \right)$. In fact all the conditions are easily checked. By Remark 5 we have

$$\frac{-D\gamma_{i+1}(x)}{\rho(-D\gamma_{i+1}(x))} \in E.$$

Let us prove, for example, that $\gamma_{i+1}(x) \neq 0$ if $x \neq x_{i+1}$. Assume for the sake of contradiction that there exists $y \neq x_{i+1}$ with $\gamma_{i+1}(y) = 0$. We would deduce that $\gamma_{i+1} \equiv 0$ on the half line $x_{i+1} + t(y - x_{i+1})$, $t \geq 0$, which contradicts, Ω being bounded, the inclusion $\{x \in \mathbb{R}^n : \gamma_{i+1}(x) < 1\} \subset \Omega$.

Since the measure is upper semicontinuous (in fact even continuous), cf. Proposition 14, with respect to the type of convergence under consideration we have the result.

Observe that, as wished, $\Omega_1 = \Omega$ if Ω is convex and $\frac{-\nu}{\rho(-\nu)} \in E$, a.e. on $\partial\Omega$ (because choosing ω the gauge of Ω centered at x_1 , we would have $\omega \in \Gamma(x_1, \Omega)$).

Since we have, with this procedure, exhausted all elements of the sequence $\{x^N\}$, we have indeed

$$\text{meas} \left[\Omega \setminus \bigcup_{i=1}^{\infty} \overline{\Omega}_i \right] = 0.$$

Example 13. Consider the case $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$, $u = u(x_1, x_2)$ and

$$\begin{cases} \left(\left(\frac{\partial u}{\partial x_1} \right)^2 - 1 \right)^2 + \left(\left(\frac{\partial u}{\partial x_2} \right)^2 - 1 \right)^2 = 0 & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Choosing the grid sequence $\{x^N\}_{N=1}^{\infty}$ in a suitable way, starting with $x^1 = (0, 0)$, we find with our procedure

$$\Omega^1 = \{x \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\} \text{ and } u(x_1, x_2) = 1 - |x_1| - |x_2| \text{ in } \Omega_1.$$

Similarly for Ω_i . Our construction is compatible with the numerical computations of [4].

We end up with an elementary convergence result that we used above.

Proposition 14. Let $\{\gamma^s\}_{s \in \mathbb{N}}$ and γ^∞ be measurable functions defined on a bounded measurable set $\Omega \subset \mathbb{R}^n$. Let

$$\begin{aligned} \Omega^s &= \{x \in \Omega : \gamma^s(x) \leq 1\}, \\ \Omega^\infty &= \{x \in \Omega : \gamma^\infty(x) \leq 1\}. \end{aligned}$$

If $\gamma^s \rightarrow \gamma^\infty$ a.e. in Ω , then

$$\text{meas}(\Omega^\infty) \geq \limsup_{s \rightarrow \infty} \text{meas}(\Omega^s).$$

If, in addition, γ^s and γ^∞ are gauges centered at $x_0 \in \Omega$ and Ω is open, then

$$\text{meas}(\Omega^\infty) = \lim_{s \rightarrow \infty} \text{meas}(\Omega^s).$$

Remark 15. Note that if γ^s and γ^∞ are merely convex, then continuity does not hold, as the following example shows. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set containing the unit ball B_1 . If

$$\gamma^s(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ \frac{1}{s}|x| + \frac{s-1}{s} & \text{if } |x| > 1, \end{cases}$$

then $\Omega^s = B_1$ for every $s \in \mathbb{N}$, while $\Omega^\infty = \Omega$.

Proof. 1) Define

$$\chi^s(x) = \begin{cases} 0 & \text{if } x \in \Omega^s, \\ 1 & \text{if } x \notin \Omega^s \end{cases}$$

and similarly for χ^∞ . Note that because of the convergence of γ^s to γ^∞ , we have that, at almost all points where $\chi^\infty(x) = 1$ (i.e. $\gamma^\infty(x) > 1$) and for large enough s , $\chi^s(x) = 1$ and thus

$$\lim_{s \rightarrow \infty} \chi^s(x) = \chi^\infty(x), \text{ a.e. } x \notin \Omega^\infty.$$

Moreover, trivially, $\liminf_{s \rightarrow \infty} \chi^s(x) \geq \chi^\infty(x) = 0$, a.e. $x \in \Omega^\infty$ and therefore

$$\liminf_{s \rightarrow \infty} \chi^s(x) \geq \chi^\infty(x), \text{ a.e. } x \in \Omega.$$

Therefore by Fatou's lemma

$$\begin{aligned} \liminf_{s \rightarrow \infty} [\text{meas}(\Omega) - \text{meas}(\Omega^s)] &= \liminf_{s \rightarrow \infty} \int_{\Omega} \chi^s(x) dx \\ &\geq \int_{\Omega} \chi^\infty(x) dx = \text{meas}(\Omega) - \text{meas}(\Omega^\infty) \end{aligned}$$

which gives the upper semicontinuity.

2) Let $B_\varepsilon = \{x \in \mathbb{R}^n : |x| \leq \varepsilon\}$ and for $A \subset \mathbb{R}^n$ define

$$A + B_\varepsilon = \{x \in \mathbb{R}^n : x = y + z \text{ with } y \in A \text{ and } |z| \leq \varepsilon\}.$$

The Hausdorff distance between two sets is then defined as

$$d(A, B) = \inf \{\varepsilon \geq 0 : A \subset B + B_\varepsilon, B \subset A + B_\varepsilon\}.$$

Observe (see below) that since γ^s and γ^∞ are gauges then

$$(16) \quad d(\Omega^s, \Omega^\infty) \rightarrow 0, \text{ as } s \rightarrow \infty,$$

and therefore (see Theorem 6.2.17 in [13])

$$\text{meas}(\Omega^\infty) = \lim_{s \rightarrow \infty} \text{meas}(\Omega^s).$$

We now establish (16). We will prove that for every $\varepsilon > 0$ we can find s sufficiently large so that

$$(17) \quad \Omega^\infty \subset \Omega^s + B_\varepsilon, \quad \Omega^s \subset \Omega^\infty + B_\varepsilon.$$

Assume without loss of generality that $x_0 = 0$. Since Ω is bounded and γ^s are gauges that converge almost everywhere to a gauge γ^∞ , the convergence is, in fact, uniform. Furthermore there exist $m, M > 0$ so that

$$m|x| \leq \gamma^s(x), \gamma^\infty(x) \leq M|x|, \forall x \in \Omega,$$

and, for s sufficiently large,

$$|\gamma^s(x) - \gamma^\infty(x)| \leq \varepsilon^2, \forall x \in \Omega.$$

Let $x \in \Omega^\infty$, i.e. $\gamma^\infty(x) \leq 1$, and choose $\delta > 0$ such that

$$\frac{\varepsilon^2}{1 + \varepsilon^2} \leq \delta \leq m\varepsilon$$

and observe that

$$\gamma^s((1 - \delta)x) = (1 - \delta)\gamma^s(x) \leq (1 - \delta)(\gamma^\infty(x) + \varepsilon^2) \leq (1 - \delta)(1 + \varepsilon^2) \leq 1,$$

$$|\delta x| \leq \delta \frac{\gamma^\infty(x)}{m} \leq \frac{\delta}{m} \leq \varepsilon.$$

Therefore $x = (1 - \delta)x + \delta x \in \Omega^s + B_\varepsilon$ which is the first inclusion in (17). The second one being proved in a similar manner, we have the claim. $\square$

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